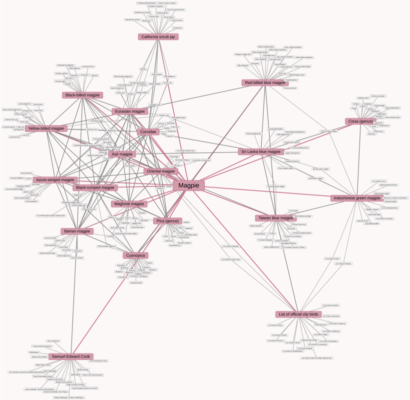
# Games, graphs, and machines

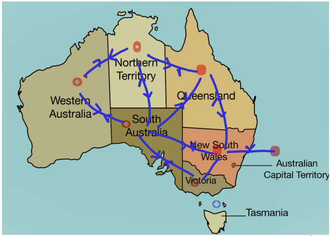



August 13, 2024

# Neighbour graph

Draw the graph whose

- vertices are the states or territories of Australia,
- two vertices are joined by an edge if they share a border.

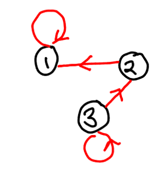



Write the adjacency matrix.

$$\begin{array}{c|cccccccc} & WA & NT & Q & NSW & SA & VIC & ACT & TAS \\ \hline WA & 0 & 1 & 0 & \cdots & & & & \\ NT & 1 & 0 & 1 & & & & & \\ Q & 0 & 1 & 0 & & & & & \\ NSW & & \vdots & & \ddots & & & & \\ SA & & \vdots & & & & & & \\ VIC & & \vdots & & & & & & \\ ACT & & & & & & & & \\ TAS & & & & & & & & \end{array}$$

# Another adjacency matrix

Write the adjacency matrix of the following directed graph.




| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| 3 | 0 | 1 | 1 |

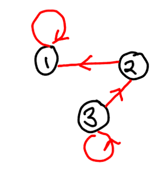
## Degree of a vertex

- The *out-degree* of a vertex is the number of edges going out of it.
- The *in-degree* of a vertex is the number of edges coming into it.

Total in deg = Total out deg = # edges.

1. Find the incoming and outgoing degrees in the previous graph.

① In: 2
Out: 1




② In: 1
Out: 1

③ In: 1
Out: 2

2. How are you read off the degrees from the adjacency matrix?

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

edges out of ①

edges to ①

# Matrix multiplication

Multiply the following matrices

$$\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 3 & 1\end{pmatrix}.$$

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 3 & 3 \\ 3 & 1 \end{pmatrix} \end{array}$$

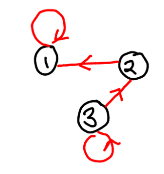
Try: $\begin{pmatrix}0 & 1\\ 3 & 1\end{pmatrix}\cdot\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & \cdots\\ \cdots & \cdots\end{pmatrix}$

# Powers of the adjacency matrix

Let $A$ be the adjacency matrix of

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = A$$

$k = 1$




$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$k = 2$

Calculate $A^2$ and $A^3$.

$$A^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$k = 3$

$$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

$$A^3 = A^2 \cdot A$$
$$= A \cdot A^2$$

$$\underbrace{A \cdot A} \cdot A$$
$$A \cdot \underbrace{A \cdot A}$$

$$A \cdot (B \cdot C)$$
$$= (A \cdot B) \cdot C$$

$3 \to 3 \to 2 \to 1$

$3 \to 2 \to 1 \to 1$

$3 \to 3 \to 3 \to 2 \to 1$

$3 \to 3 \to 2 \to 1 \to 1$

$3 \to 2 \to 1 \to 1 \to 1$

# Why does $A^k$ count length $k$ paths?

**Theorem**
*The $(i, j)$ entry if $A^k$ is the number of paths from vertex $i$ to vertex $j$.*

Suppose $n = 3$.

$$A^2_{i,j} = A_{i,1} \cdot A_{1,j} + A_{i,2} \cdot A_{2,j} + A_{i,3} \cdot A_{3,j}$$

Tomorrow

# Why does $A^k$ count length $k$ paths?

We have $n = 3$.

$$A^3_{i,j} = A^2_{i,1} \cdot A_{1,j} + A^2_{i,2} \cdot A_{2,j} + A^2_{i,3} \cdot A_{3,j}$$

# Why does $A^k$ count length $k$ paths?

We have $n = 3$.

$$A^4_{i,j} = A^3_{i,1} \cdot A_{1,j} + A^3_{i,2} \cdot A_{2,j} + A^3_{i,3} \cdot A_{3,j}$$